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Journal of Approximation Theory 126 (2004) 218–232
Journal of Approximation Theory
http://www.elsevier.com/locate/jat

On a problem of H. Shapiro

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Received 23 September 2003; accepted in revised form 1 December 2003

Communicated by Paul Nevai

Abstract

Let μ be a real measure on the line such that its Poisson integral $M(z)$ converges and satisfies

$$|M(x+iy)| \leqslant Ae^{-cy^{\alpha}}, \quad y \to +\infty,$$

for some constants $A, c > 0$ and $0 < \alpha \leqslant 1$. We show that for $1/2 < \alpha \leqslant 1$ the measure μ must have many sign changes on both positive and negative rays. For $0 < \alpha \leqslant 1/2$ this is true for at least one of the rays, and not always true for both rays. Asymptotical bounds for the number of sign changes are given which are sharp in some sense.

MSC: 30D99; 42A38

Keywords: Sign changes; Poisson integral; Oscillations

1. Introduction

Let f be a real-valued function from $L_\infty(\mathbf{R})$ and let

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(t)\,dt}{(x-t)^2+y^2}, \quad z = x+iy \in \mathbf{C}\backslash\mathbf{R},$$

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2003.12.003

be its Poisson integral. Shapiro [7] asks how many sign changes a real even function f must have if its Poisson integral satisfies

$$F(iy) = O(e^{-cy^{\alpha}}), \quad y \to +\infty, \tag{1}$$

for some $c > 0$ and $0 < \alpha \leqslant 1$. As shown in [6, Theorem L] and [5, Theorem 7.6.3], for $\alpha = 1$ and an even function f condition (1) is equivalent to

$$|F(z)| \leqslant Ae^{-c|y|}, \quad z = x + iy \in \mathbf{C} \setminus \mathbf{R}, \tag{2}$$

where $A > 0$ is independent of x and y. Condition (2) is of interest because [5, Theorem 7.6.3] it is equivalent to the condition that the spectrum of f (i.e. the support of its Fourier transform) is disjoint from $(-c, c)$.

The following phenomenon has been known for a long time: if a real function (or more generally: measure, distribution) has a spectral gap at the origin then it must have many sign changes. This phenomenon has been deeply studied in the recent work by Eremenko and Novikov [1]. In [4], it has been established that a similar phenomenon occurs when the Fourier transform is real analytic in a neighborhood of the origin but not on the whole real line.

We shall consider the following question: Let $\mu \not\equiv 0$ be a real Borel measure on $\mathbf{R}$ such that

$$\int_{-\infty}^{\infty} \frac{d|\mu|(t)}{1+t^2} < \infty, \tag{3}$$

and let

$$M(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\, d\mu(t)}{(x-t)^2 + y^2}, \quad z = x + iy \in \mathbf{C} \setminus \mathbf{R},$$

be its Poisson integral. Assume there are positive constants c, A, q and a constant $0 < \alpha \leqslant 1$ such that

$$|M(z)| \leqslant Ae^{-c|y|^{\alpha}}, \quad \text{for } |y| \geqslant q, \quad z = x + iy. \tag{4}$$

How many sign changes must the measure μ have?

To make this question precise, let us introduce counting functions for the sign changes of a real measure. Let μ be a locally finite real Borel measure on $\mathbf{R}$ and let $[a, b)$ be a finite half-interval. Let J be a partition of $[a, b)$, that is a finite set of points $\{x_1, x_2, \ldots, x_n\}$ such that $a < x_1 < x_2 < \cdots < x_n < b$. Consider the finite sequence

$$\mu([a, x_1)), \mu([x_1, x_2)), \ldots, \mu([x_n, b)),$$

and denote by v_J the number of its sign changes. We define the number of sign changes of μ on $[a, b)$ as follows:

$$v([a, b)) = \sup_J v_J,$$

where sup is taken over all partitions J of $[a, b)$. Clearly, $v([a, b))$ is either a non-negative integer or $+\infty$, and in the first case the sup is attained. Observe also that $v([a, b))$ is a non-decreasing function of $[a, b)$. For $t > 0$ we set

$$n_+(t) = v([0, t)), \quad n(t) = v([-t, t)). \tag{5}$$

These functions are non-negative, non-decreasing and integer-valued (in extended sense).

2. Results

Our approach is based on some ideas of Levin [3, pp. 403–404] and also an idea of Kahane [2, pp. 76–77].

The following result is due to Levin [3, Appendix II, Theorem 5]: *Let $\mu \not\equiv 0$ be a real finite Borel measure on the real line whose spectrum is disjoint from $(-c, c)$. Then*

$$\liminf_{R\to\infty}\left\{\int_0^R \frac{n(t)}{t}\,dt - \frac{2c}{\pi}R\right\} > -\infty.$$

Observe that there is no non-trivial function F satisfying (4) with $\alpha > 1$. This follows, for example, from the mentioned Theorem 7.6.3 in [5]. We shall consider the cases $\alpha = 1$ and $0 < \alpha < 1$ separately. Our first result extends Levin's theorem to measures satisfying (3), and also gives one-sided estimates on the number of sign changes:

Theorem 1. *Let $\mu \not\equiv 0$ be a real Borel measure on* **R** *satisfying* (3). *If its Poisson integral satisfies condition* (4) *with $\alpha = 1$, then*:

(i) $$\liminf_{R\to\infty}\left\{\int_1^R \left(\frac{1}{t^2} + \frac{1}{R^2}\right) n_+(t)\,dt - \frac{c}{\pi}\log R\right\} > 0;$$

(ii) $$\liminf_{R\to\infty}\left\{\int_1^R \frac{n(t)}{t}\,dt - \frac{2c}{\pi}R + 3\log R\right\} > 0.$$

Corollary. *Let μ be a real Borel measure on* **R** *satisfying* (3). *If its spectrum is disjoint from $(-c, c)$, then the assertion of Theorem* 1 *holds.*

This corollary has been announced without proof in [4] and later extended in [1] (with a bit less precise estimates of the asymptotical behavior of n_+ and n) for much more general class of measures (and distributions).

Our next result extends the part (ii) of Theorem 1 to the case $\alpha < 1$:

Theorem 2. *Let $\mu \not\equiv 0$ be a real Borel measure on* **R** *satisfying* (3). *If its Poisson integral satisfies condition* (4) *with $0 < \alpha < 1$, then*

$$\liminf_{R\to\infty}\left\{\int_1^R \frac{n(t)}{t}\,dt - \frac{c\,\Gamma((1+\alpha)/2)}{\sqrt{\pi}\,\Gamma(1+\alpha/2)}R^{\alpha} + 3\log R\right\} > 0.$$

One may ask if the measures whose Poisson integral satisfies (4) with $\alpha<1$ must have many sign changes on both half-lines $(-\infty,0)$ and $(0,\infty)$. Our Example 3 in the next section shows that the answer is negative for $0<\alpha\leqslant 1/2$. It means that for these values of α there is no analogue of the assertion (i) of Theorem 1. However, if $1/2<\alpha<1$ then such an estimate is possible:

Theorem 3. *Let* $\mu\not\equiv 0$ *be a real Borel measure on* $\mathbf{R}$ *satisfying* (3). *If its Poisson integral satisfies condition* (4) *with* $1/2<\alpha<1$, *then*

$$\liminf_{R\to\infty}\frac{1}{\log R}\left\{\int_1^R\left(\frac{1}{t^{\alpha+1}}+\frac{t^{\alpha-1}}{R^{2\alpha}}\right)n_+(t)\,dt-[\sin(\pi/(2\alpha))]^{\alpha}c\log R\right\}>-\infty.$$

3. Sharpness of Theorems 1–3

Example 1 (Sharpness of Theorem 1). Let μ be an absolutely continuous measure with the density $\sin ct$. Then a direct calculation shows that

$$M(z)=(\operatorname{sgn} y)e^{-c|y|}\sin cx,\quad z\in\mathbf{C}\backslash\mathbf{R};$$

$$n_+(t)=(c/\pi)t+O(1),\quad n(t)=2(c/\pi)t+O(1),\quad t\to\infty.$$

The following example is similar to the example in [3], p. 410.

Example 2 (Sharpness of Theorems 2 and 3). Let $0<\alpha<1$. Set

$$f_\alpha(z)=\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^{2/\alpha}}\right),\quad 0<\alpha<1.$$

Standard arguments show that, for $\Im(re^{i\theta})\geqslant 1$,

$$\log|f_\alpha(re^{i\theta})|=\left(\frac{\pi\cos(\alpha(\theta-\pi/2))}{\sin(\alpha\pi/2)}\right)r^{\alpha}+O(\log r),\quad r\to\infty, \tag{6}$$

and (cf. [3, p. 196])

$$\log|f'_\alpha(\pm k^{1/\alpha})|=(\pi\cot(\pi\alpha/2))k(1+o(1)),\quad k\to\infty.$$

These relations imply that the following representation holds:

$$\frac{1}{f_\alpha(z)}=\sum_{k=1}^{\infty}\left(\frac{1}{f'_\alpha(k^{1/\alpha})(z-k^{1/\alpha})}+\frac{1}{f'_\alpha(-k^{1/\alpha})(z+k^{1/\alpha})}\right).$$

Let us introduce the atomic measure

$$\mu_\alpha = \pi \sum_{k=1}^{\infty} \left(\frac{1}{f'_\alpha(k^{1/\alpha})} \delta_{k^{1/\alpha}} + \frac{1}{f'_\alpha(-k^{1/\alpha})} \delta_{-k^{1/\alpha}} \right).$$

Evidently,

$$\Im\left(\frac{1}{f_\alpha(z)}\right) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y \, d\mu_\alpha(t)}{(x-t)^2 + y^2}$$

is the Poisson integral of the measure μ_α.

Since

$$\operatorname{sgn} f'_\alpha(k^\alpha \operatorname{sgn} k) = (-1)^k \operatorname{sgn} k, \quad k = \pm 1, \pm 2, \ldots,$$

then

$$n_+(t) = t^\alpha + O(1), \quad n(t) = 2t^\alpha + O(1), \quad t \to \infty,$$

and, by (6),

$$\left| \Im \frac{1}{f_\alpha(z)} \right| \leqslant \exp[-(\pi \cot(\pi\alpha))|y|^\alpha + O(\log |y|)], \quad \text{for } |y| \geqslant 1,$$

we see that the inequalities of Theorems 2 and 3 are sharp in the sense of order. We do not know whether the coefficients of R^α and $\log R$ are the best possible.

The following example shows that there exist measures μ satisfying (3) and (4) with $0 < \alpha \leqslant 1/2$ such that μ is positive on a half-line.

Example 3. Let $0 < \alpha < 1/2$ and

$$g_\alpha(z) = \prod_{k=1}^{\infty} \left(1 + \frac{z}{k^{1/\alpha}} \right).$$

Then

$$\log |g_\alpha(re^{i\theta})| = \frac{\pi \cos \alpha\theta}{\sin \pi\alpha} r^\alpha + O(\log r), \quad r \to \infty, \ |\Im(re^{i\theta})| \geqslant 1. \tag{7}$$

and

$$\log |g'_\alpha(-k^{1/\alpha})| = (\pi \cot \pi\alpha) k (1 + o(1)), \quad k \to \infty.$$

Then

$$\frac{1}{g_\alpha(z)} = \sum_{k=1}^{\infty} \frac{1}{g'(-k^{1/\alpha})(z + k^{1/\alpha})},$$

and

$$\Im \frac{1}{g_\alpha(z)} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\, d\mu_\alpha(t)}{(x-t)^2+y^2}$$

is the Poisson integral of the measure

$$\mu_\alpha = \pi \sum_{k=1}^{\infty} \frac{1}{g'(-k^{1/\alpha})}\,\delta_{-k^{1/\alpha}}.$$

Evidently, (7) implies

$$\left|\Im \frac{1}{g_\alpha(z)}\right| \leqslant \exp\left(\frac{\pi}{2\sin(\pi\alpha/2)}|y|^\alpha + O(\log y)\right) \quad \text{for } |y|\geqslant 1,$$

nevertheless, $n_+(t) \equiv 0$.

In the case $\alpha = 1/2$ we replace g_α with $(1+z)\cosh\sqrt{z}$.

4. Proof of Theorem 1

We assume that a real measure μ satisfies conditions (3) and (4) with $\alpha = 1$.

The following function analytic in $\mathbf{C}\backslash\mathbf{R}$ will play an important role:

$$G(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t).$$

Evidently, it satisfies

$$\overline{G(z)} = G(\bar{z}) \tag{8}$$

and

$$\Im G(z) = M(z), \quad z\in\mathbf{C}\backslash\mathbf{R}. \tag{9}$$

Lemma 1. *The estimate holds*:

$$|G(z)| \leqslant A\,\frac{|z|^2+1}{|z|}, \quad z\in\mathbf{C}\backslash\mathbf{R}.$$

(*Here and further we denote by A a positive constant not necessary the same everywhere*.)

Proof. Clearly, G is a difference of two functions analytic in $\mathbf{C}\backslash\mathbf{R}$ and having positive (negative) imaginary part in the upper (lower) half-plane. The assertion of Lemma 1 follows immediately from the well-known Caratheodory inequality (see, e.g. [3, p. 18]. □

Lemma 2. *The estimate holds*:

$$|G'(z)| \leqslant Ae^{-c|y|} \quad \textit{for } |y| \geqslant 2q. \tag{10}$$

Proof. By (9) and the Schwarz formula we have

$$G(z+\zeta) = \frac{i}{2\pi}\int_0^{2\pi} M(z+qe^{i\theta})\frac{qe^{i\theta}+\zeta}{qe^{i\theta}-\zeta}\,d\theta + \Re G(z) \quad \text{for } |\Im z| \geqslant 2q,\ |\zeta| < q.$$

Differentiating with respect to ζ and then setting $\zeta = 0$, we obtain

$$G'(z) = \frac{i}{\pi q}\int_0^{2\pi} M(z+qe^{i\theta})e^{-i\theta}\,d\theta \quad \text{for } |\Im z| \geqslant 2q.$$

Hence

$$|G'(z)| \leqslant \frac{2}{q}\max_{0\leqslant\theta\leqslant 2\pi} |M(z+qe^{i\theta})|,$$

and condition (4) with $\alpha = 1$ implies (10). □

Lemma 3. *There exists a real constant D such that the estimate holds*:

$$|G(z) - D| \leqslant Ae^{-c|y|} \quad \textit{for } |y| \geqslant 2q. \tag{11}$$

Proof. Let us define, for $\Im z > 0$,

$$H(z) = \int_z^{\infty} G'(\zeta)\,d\zeta,$$

where the integral is taken along the vertical line going upwards from z. Using (10), it is easy to see that the integral is an analytic function in the upper half-plane and satisfies

$$|H(z)| \leqslant Ae^{-c|y|} \quad \text{for } |y| \geqslant 2q. \tag{12}$$

For $\Im z < 0$ we set $H(z) = \overline{H(\bar{z})}$. The function H is analytic in the lower half-plane and satisfies (12). Since $[G+H]' \equiv 0$, we see that $G+H$ is a constant D_+ (D_-), say, in the upper (lower) half-plane. Since $\Im[G+H](iy) = [M+\Im H](iy)$ tends to zero as $|y| \to \infty$, the constants $D_\pm$ are real. Since $\Re[G+H](iy) = \Re[G+H](-iy)$, we conclude that $D_+ = D_-$. □

Corollary. *Function G is not constant.*

Proof. Since $\Im G = M \not\equiv 0$, G cannot be a real constant. On the other hand, Lemma 3 shows that $G(iy)$ tends to real constant D as $y \to \infty$. □

Lemma 4. *The support of the measure* μ *is unbounded from right and left.*

Proof. Assume, that $\operatorname{supp}\mu \subset (-\infty, d)$, $d < +\infty$, Noting that

$$\frac{1}{t-z} - \frac{t}{1+t^2} = \left(z + \frac{z^2+1}{t-z}\right)\frac{1}{1+t^2}, \tag{13}$$

and using condition (3), we see that

$$|G(z)| \leqslant \int_{-\infty}^{d} \left(|z| + \frac{|z|^2+1}{|t-z|}\right)\frac{d|\mu|(t)}{1+t^2} \leqslant A(|z|^2+1) \quad \text{for } \Re z \geqslant 2d. \tag{14}$$

This bound and (11) show that the well-known Carlson theorem (see, e.g. [8, Section 5.8]) is applicable to $G - D$ and hence $G \equiv D$. Nevertheless, $\Im G = M \not\equiv 0$, and we obtain a contradiction. □

Let us introduce the sequence of atomic measures

$$\mu_p = \sum_{k=-\infty}^{\infty} \mu([k2^{-p}, (k+1)2^{-p}))\delta_{k2^{-p}}, \quad p = 1, 2, \ldots,$$

where δ_a denotes the unit measure at point a. According to Lemma 3, each measure μ_p has support unbounded from right and left. Condition (3) implies

$$\sup_{p \geqslant 1} \int_{-\infty}^{\infty} \frac{d|\mu_p|(t)}{1+t^2} < \infty. \tag{15}$$

Let us define the sequence of meromorphic in **C** functions

$$G_p(z) = \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu_p(t), \quad p = 1, 2, \ldots.$$

Each function G_p takes real values on **R** and its poles are real and simple; the set of the poles is unbounded from right and left. Note that, for any real constant A, function $G_p - A$ has a zero between any two consecutive poles having residues of the same sign.

The following lemma concerns convergence of the sequence $\{G_p\}$ as $p \to \infty$.

Lemma 5. (i) *On any compact set* K *lying entirely in the upper or lower half-plane the sequence* G_p *tends to* G *uniformly as* $p \to \infty$.

(ii) *On any compact set in* **C** *the following estimate holds*:

$$|G_p(z)| \leqslant A|y|^{-1},$$

where A *is independent of* z *and* p.

Proof. (i) We have

$$G(z) - G_p(z)$$
$$= \sum_{k=-\infty}^{\infty} \int_{k2^{-p}}^{(k+1)2^{-p}} \left(\frac{1}{t-z} - \frac{1}{k2^{-p}-z} - \frac{t}{1+t^2} + \frac{k2^{-p}}{1+(k2^{-p})^2} \right) d\mu(t)$$
$$= -\sum_{k=-\infty}^{\infty} \int_{k2^{-p}}^{(k+1)2^{-p}} \left[\int_{k2^{-p}}^{t} \left(\frac{1}{(u-z)^2} - \frac{1-u^2}{(1+u^2)^2} \right) du \right] d\mu(t).$$

For z belonging to a fixed compact set lying entirely in the upper or lower half-plane the following inequality holds:

$$\left| \frac{1}{(u-z)^2} \right| \leqslant \frac{1}{1+u^2} \max_{u \in \boldsymbol{R}} \left| \frac{1+u^2}{(u-z)^2} \right| \leqslant \frac{A}{1+u^2},$$

where A is independent of z and u. Therefore

$$|G(z) - G_p(z)| \leqslant \frac{A}{2^p} \int_{-\infty}^{\infty} \frac{d|\mu_p|(u)}{1+u^2}.$$

Using (14), we conclude that on the compact set

$$|G(z) - G_p(z)| \leqslant A2^{-p} \to 0 \quad \text{as } p \to \infty.$$

(ii) Observe that (13) implies

$$\left| \frac{1}{t-z} - \frac{t}{1+t^2} \right| \leqslant \left(|z| + \frac{|z|^2+1}{|y|} \right) \frac{1}{1+t^2}.$$

Hence, for z belonging to a compact set in $\mathbf{C}$,

$$|G_p(z)| \leqslant \frac{A}{|y|} \int_{-\infty}^{\infty} \frac{d|\mu_p|(t)}{1+t^2},$$

where A is independent of z and p. By condition (14) the integral in the right hand side is bounded by a constant independent of p. □

Let $\eta > 0$ be a number such that $G(i\eta) - D \neq 0$, where D is the constant from Lemma 3. Such η exists in virtue of corollary to Lemma 3. Set

$$f(z) = G(z + i\eta) - D, \quad f_p(z) = G_p(z + i\eta) - D, \quad p = 1, 2, \ldots$$

Choose $\varepsilon \in (0, \eta)$ so small that f does not vanish in the closed disc $\{z: |z| \leqslant \varepsilon\}$. By Lemma 5(i), f_p also does not vanish in the disc for all sufficiently large p. Further we shall consider only such values of p.

Let us start with the proof of assertion (i) of Theorem 1.

Denote by $z_{k,p}$ zeros and by $\zeta_{k,p}$ poles of f_p situated in the right half-plane. We agree to enumerate $\zeta_{j,p}$, $j = 1, 2, \ldots$, in order of increasing real parts. By the Carleman formula for the right half-plane (see, e.g., [3, p. 224] where it is written for the upper half-plane) we have for any $R > \varepsilon$:

$$\sum_{|z_{k,p}|<R}\left(\Re\frac{1}{z_{k,p}}-\frac{\Re z_{k,p}}{R^2}\right)-\sum_{|\zeta_{k,p}|<R}\left(\Re\frac{1}{\zeta_{k,p}}-\frac{\Re\zeta_{k,p}}{R^2}\right)$$
$$=\frac{1}{\pi R}\int_{-\pi/2}^{\pi/2}\log|f_p(Re^{i\theta})\cos\theta\,d\theta+\frac{1}{2\pi}\int_{\varepsilon}^{R}\left(\frac{1}{t^2}-\frac{1}{R^2}\right)\log|f_p(it)f_p(-it)\,dt$$
$$+\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\Re\left[\log f_p(\varepsilon e^{i\theta})\left(\frac{e^{-i\theta}}{\varepsilon}+\frac{\varepsilon e^{i\theta}}{R^2}\right)\right]d\theta. \tag{16}$$

Let us take lim sup as $p \to \infty$.

To do this in the right-hand side of (16) we note, by Lemma 5(ii), that the following inequality holds on any compact set:

$$\log|f_p(z)| \leqslant \log\frac{A}{|y+\eta|}.$$

Taking into account Lemma 5(i) and the Fatou lemma, we obtain that

$$\limsup_{p\to\infty}\ \{\text{the right-hand side of (16)}\}$$
$$\leqslant\frac{1}{\pi R}\int_{-\pi/2}^{\pi/2}\log|f(Re^{i\theta})|\cos\theta\,d\theta+\frac{1}{2\pi}\int_{\varepsilon}^{R}\left(\frac{1}{t^2}-\frac{1}{R^2}\right)\log|f(it)f(-it)|\,dt$$
$$+\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\Re\left[\log f(\varepsilon e^{i\theta})\left(\frac{e^{-i\theta}}{\varepsilon}+\frac{\varepsilon e^{i\theta}}{R^2}\right)\right]d\theta.$$

Using Lemmas 1 and 3, we conclude that

$$\limsup_{p\to\infty}\{\text{the right-hand side of (16)}\}$$
$$\leqslant -\frac{c}{\pi}\log R+O(1),\quad \text{as } R\to\infty. \tag{17}$$

Let us now estimate from below the left-hand side of (16). Note that the poles $\zeta_{j,p}$ of f_p are simple and situated on the line $\{z\colon \Im z=-\eta\}$, and between any two consecutive poles having residues of the same sign there is at least one zero of f_p on the same line. Let us denote by Q_p the set of all poles $\zeta_{j,p}$ such that the nearest pole from the right has residue of opposite sign. If $\zeta_{j,p}=\xi_{j,p}-i\eta\notin Q_p$, then there is a zero $z_{k(j),p}=x_{k(j),p}-i\eta$ such that the 'interlacing condition' holds

$$\xi_{j,p}<x_{k(j),p}<\xi_{j+1,p}. \tag{18}$$

Set

$$S_1^{(p)}(R)=\sum_{|z_{k,p}|<R}\left(\Re\frac{1}{z_{k,p}}-\frac{\Re z_{k,p}}{R^2}\right)-\sum_{|\zeta_{j,p}|<R;\ \zeta_{j,p}\notin Q_p}\left(\Re\frac{1}{\zeta_{j,p}}-\frac{\Re z_{j,p}}{R^2}\right),$$

$$S_2^{(p)}(R)=\sum_{|\zeta_{j,p}|<R;\ \zeta_{j,p}\in Q_p}\left(\Re\frac{1}{\zeta_{j,p}}-\frac{\Re\zeta_{j,p}}{R^2}\right),$$

so that the left-hand side of (16) is $S_1^{(p)}(R) - S_2^{(p)}(R)$. Observe that

$$\begin{aligned}&\limsup_{p\to\infty}\ \{\text{the left-hand side of}(16)\}\\&\quad=\limsup_{p\to\infty}\ (S_1^{(p)}(R) - S_2^{(p)})(R) \geqslant \limsup_{p\to\infty}\ S_1^{(p)}(R) - \limsup_{p\to\infty}\ S_2^{(p)}(R).\end{aligned} \tag{19}$$

Let us first estimate $S_1^{(p)}(R)$ from below. To this end we omit all summands which correspond to the zeros $z_{k,p}$ being not $z_{k(j),p}$. Then we get

$$\begin{aligned}S_1^{(p)}(R) &\geqslant \sum_{|z_{k(j),p}|<R;\ \zeta_{j,p}\notin Q_p}\left(\Re\frac{1}{z_{k(j),p}} - \frac{\Re z_{k(j),p}}{R^2}\right) - \sum_{|\zeta_{j,p}|<R;\ \zeta j,p\notin Q_p}\left(\Re\frac{1}{\zeta_{j,p}} - \frac{\Re\zeta_{j,p}}{R^2}\right)\\&= \sum_{|z_{k(j),p}|<R;\ \zeta_{j,p}\notin Q_p}\left(\frac{x_{k(j),p}}{x_{k(j),p}^2+\eta^2} - \frac{x_{k(j),p}}{R^2}\right) - \sum_{|\zeta_{j,p}|<R;\ \zeta_{j,p}\notin Q_p}\left(\frac{\xi_{j,p}}{\xi_{j,p}^2+\eta^2} - \frac{\xi_{j,p}}{R^2}\right)\\&= \int_0^{\sqrt{R^2-\eta^2}}\left(\frac{t}{t^2+\eta^2} - \frac{t}{R^2}\right)d(b_1(t) - b_2(t))\\&= \int_0^{\sqrt{R^2-\eta^2}}\left(\frac{t^2-\eta^2}{(t^2+\eta^2)^2} + \frac{1}{R^2}\right)(b_1(t) - b_2(t))\,dt,\end{aligned}$$

where

$$b_1(t) = \#\{x_{k(j),p}\colon\ x_{k(j),p}<t,\ \zeta_{j,p}\notin Q_p\};\quad b_2(t) = \#\{\xi_{j,p}\colon\ \xi_{j,p}<t,\ \zeta_{j,p}\notin Q_p\}.$$

The ‘interlacing condition’ (18) implies that

$$|b_1(t) - b_2(t)| \leqslant 1.$$

Thus,

$$\limsup_{p\to\infty}\ S_1^{(p)}(R) \geqslant -A > -\infty, \tag{20}$$

where A does not depend on R.

Now, let us estimate $S_2^{(p)}(R)$ from above. Set

$$b_3(t) = \#\{\xi_{j,p}\colon\ 0<\xi_{j,p}<t,\ \zeta_p\in Q_p\},\quad t>0.$$

Then we have

$$\begin{aligned}S_2^{(p)}(R) &= \int_0^{\sqrt{R^2-\eta^2}}\left(\frac{t}{t^2+\eta^2} - \frac{t}{R^2}\right)db_3(t)\\&= \int_0^{\sqrt{R^2-\eta^2}}\left(\frac{t^2-\eta^2}{(t^2+\eta^2)^2} + \frac{1}{R^2}\right)b_3(t)\,dt\\&\leqslant \int_0^{\sqrt{R^2-\eta^2}}\left(\frac{1}{t^2+\eta^2} + \frac{1}{R^2}\right)b_3(t)\,dt.\end{aligned}$$

Recall that the residue at pole $\zeta_{j,p}$ is $\mu([j2^{-p},(j+1)2^{-p}))$ and the set Q_p consists of poles $\zeta_{j,p}$, for which the next pole from the right has residue of opposite sign. Hence, it follows that $b_3(t) \leqslant s_j + 1$ where s_j is the number of sign changes in the sequence

$$\mu([0,2^{-p})), \mu([2^{-p}, 2\cdot 2^{-p})), \dots, \mu([j2^{-p},(j+1)2^{-p})),$$

where j is the greatest integer such that $j2^{-p} < t$. Evidently, $s_j \leqslant n_+(t)$, where n_+ is defined by (5), and we have

$$b_3(t) \leqslant n_+(t) + 1.$$

Hence

$$\begin{aligned}\limsup_{p\to\infty} S_2^{(p)}(R) &\leqslant \int_0^{\sqrt{R^2-\eta^2}} \left(\frac{1}{t^2+\eta^2} + \frac{1}{R^2}\right)(n_+(t)+1)\,dt \\ &\leqslant \int_1^R \left(\frac{1}{t^2}+\frac{1}{R^2}\right) n_+(t)\,dt + A, \end{aligned} \tag{21}$$

where A does not depend on R (without loss of generality we assume that $n_+(t) < \infty$ for each $t > 0$, otherwise the assertion (i) of Theorem 1 is trivial).

Taking together the inequalities (17), (19), (20) and (21), we obtain the assertion (i) of Theorem 1.

The proof of assertion (ii) is similar to the proof of (i), but instead of the Carleman formula we use the Jensen formula.

Let us denote by $\{z_{k,p}\}$ the set of *all* zeros of f_p and by $\{\zeta_{j,p}\}$ the set of *all* its poles. We agree to enumerate $\zeta_{j,p}$, $-\infty < j < \infty$, in order of increasing real parts.

By the Jensen formula,

$$\sum_{|z_{k,p}|<R} \log\frac{R}{|z_{k,p}|} - \sum_{|\zeta_{j,p}|<R} \log\frac{R}{|\zeta_{j,p}|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f_p(Re^{i\theta})|\,d\theta - \log|f_p(0)|. \tag{22}$$

Let us take lim sup as $p \to \infty$.

The same arguments as in the proof of assertion (i) give

$$\begin{aligned}&\limsup_{p\to\infty} \{\text{the right-hand side of (22)}\} \\ &\quad\leqslant \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|. \end{aligned} \tag{23}$$

Hence, using Lemmas 1 and 3, we get

$$\limsup_{p\to\infty} \{\text{the right-hand side of (22)}\} \leqslant -\frac{2c}{\pi}R + O(1), \quad R\to\infty. \tag{24}$$

To estimate the left-hand side of (22) from below, we denote by Q_p the set of all (not only in the right half-plane as in the proof of assertion (i)) poles $\zeta_{j,p}$ such that the nearest pole from the right has the residue of opposite sign. Then, again, if $\zeta_{j,p} = \xi_{j,p} - i\eta \notin Q_p$, then there is a zero $z_{k(j),p} = x_{k(j),p} - i\eta$ such that (18) holds.

Denote

$$T_1^{(p)}(R) = \sum_{|z_{k,p}|<R} \log \frac{R}{|z_{k,p}|} - \sum_{|\zeta_{j,p}|<R;\ \zeta_{j,p}\notin Q_p} \log \frac{R}{|\zeta_{p,j}|};$$

$$T_2^{(p)}(R) = \sum_{|\zeta_{j,p}|<R;\ \zeta_{j,p}\in Q_p} \log \frac{R}{|\zeta_{j,p}|},$$

so that the left-hand side of (22) is $T_1^{(p)}(R) - T_2^{(p)}(R)$ and

$$\limsup_{p\to\infty} \{\text{the left-hand side of (22)}\}$$
$$\geqslant \limsup_{p\to\infty} T_1^{(p)}(R) - \limsup_{p\to\infty} T_2^{(p)}(R). \tag{25}$$

Let us estimate $T_1^{(p)}(R)$ from below. We have

$$T_1^{(p)}(R) \geqslant \sum_{|z_{k(j),p}|<R;\ \zeta_{j,p}\notin Q_p} \log \frac{R}{|z_{k(j),p}|} - \sum_{|\zeta_{j,p}|<R;\ \zeta_{j,p}\notin Q_p} \log \frac{R}{|\zeta_{j,p}|}$$
$$= \int_0^{\sqrt{R^2-\eta^2}} \log \frac{R}{\sqrt{t^2+\eta^2}}\, d(\beta_1(t)-\beta_2(t))$$
$$= \int_0^{\sqrt{R^2-\eta^2}} \frac{t}{t^2+\eta^2} (\beta_1(t)-\beta_2(t))\, dt,$$

where

$$\beta_1(t) = \#\{x_{k(j),p}\colon\ |x_{k(j),p}|<t;\ \ \zeta_{j,p}\notin Q_p\},$$
$$\beta_2(t) = \#\{\xi_{j,p}\colon\ |\xi_{j,p}|<t,\ \ \zeta_{j,p}\notin Q_p\} \quad \text{for } t>0.$$

The ‘interlacing condition’ (18) implies that

$$|\beta_1(t)-\beta_2(t)| \leqslant 2.$$

Therefore

$$\limsup_{p\to\infty} T_1^{(p)}(R) > -2\log R. \tag{26}$$

To estimate $T_2^{(p)}(R)$ from above, we set

$$\beta_3(t) = \#\{\xi_{j,p}\colon\ |\xi_{j,p}|<t,\ \zeta_{j,p}\in Q_p\}.$$

Then

$$T_2^{(p)}(R) = \int_0^{\sqrt{R^2-\eta^2}} \log \frac{R}{\sqrt{t^2+\eta^2}}\, d\beta_3(t) = \int_0^{\sqrt{R^2-\eta^2}} \frac{t}{t^2+\eta^2} \beta_3(t)\, dt.$$

As by estimation of $S_2^{(p)}(R)$ in the proof of assertion (i), we see that $\beta_3(t) \leqslant \sigma_l^j + 1$ where σ_l^j is the number of sign changes in the sequence

$$\mu([l2^{-p},(l+1)2^{-p})), \ldots, \mu([j2^{-p},(j+1)2^{-p})),$$

where l is the smallest integer such that $l2^{-p} \geqslant -t$ and j is the greatest integer such that $j2^{-p} < t$. Since $\sigma_l^j \leqslant n(t)$, where n is defined by (5), we obtain $\beta_3(t) \leqslant n(t) + 1$ and

$$\limsup_{p\to\infty} T_2^{(p)}(R) \leqslant \int_0^{\sqrt{R^2-\eta^2}} \frac{t}{t^2+\eta^2}(n(t)+1)\,dt$$
$$\leqslant \int_1^R \frac{n(t)}{t}\,dt + \log R + A, \tag{27}$$

where A is independent of R.

Joining (22), (24), (25), (26) and (27), we obtain assertion (ii). □

5. Proofs of Theorems 2 and 3

The proof of Theorem 2 is similar to that of Theorem 1. Lemmas 1, 2, 3, 5 and corollary to Lemma 3 remain in force when we replace $|y|$ with $|y|^\alpha$ in the right-hand sides of (10) and (11).

Lemma 4 remains in force only under additional condition $1/2 < \alpha < 1$. In this case it is easy to see that (12) remains in force in the angle $\{z\colon |\arg(z-2d)| \leqslant \pi/(2\alpha)\}$. Taking into account the mentioned change in (11), we see that the function $G(w^{1/\alpha} + 2d)$ satisfies conditions of the Carlson theorem in the half-plane $\{w\colon \Re w \geqslant 0\}$, and we obtain the desired contradiction. In the general case $0 < \alpha < 1$, the assertion of Lemma 4 should be replaced by the following: supp μ cannot be bounded (from both sides simultaneously). Indeed, if the support is bounded, then $G - D$ is analytic at ∞ and therefore cannot tend to zero faster than some power of $1/|z|$ without being constant.

We introduce functions f and f_p as in the proof of Theorem 1 and denote by $\{z_{k,p}\}$ and $\{\zeta_{j,p}\}$ sets of zeros and poles of f_p, respectively with the same agreement concerning enumeration of the latter set (however, if supp μ is bounded from the left (right), then $j = 1, 2, \dots,$ ($j = \cdots, -2, -1$)). Then we apply the Jensen formula (22). Evidently, (23) remains in force. Since change of bound in Lemma 3 we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta \leqslant -\frac{c\Gamma((1+\alpha)/2)}{\sqrt{\pi}\Gamma(1+\alpha/2)} R^\alpha + O(1).$$

Noting that (25), (26), (27) remain in force, we obtain the assertion of Theorem 2.

The proof of Theorem 3 differs from that of Theorem 1(i) by application of the Carleman formula for the angle $\{z\colon |\arg z| \leqslant \pi/(2\alpha)\}$ instead of the right half-plane. We apply this formula to the same function f_p and with the same meaning of notations $z_{k,p}$, $\zeta_{j,p}$ and ε. Denoting by A_R the sector

$\{z: |z| < R, |\arg z| < \pi/(2\alpha)\}$, we have

$$\sum_{z_{k,p}\in A_R}\left(\Re\left(\frac{1}{z_{k,p}^{\alpha}}\right)-\frac{\Re(z_{k,p}^{\alpha-1})}{R^{2\alpha}}\right)-\sum_{\zeta_{j,p}\in A_R}\left(\Re\left(\frac{1}{\zeta_{j,p}^{\alpha}}\right)-\frac{\Re(\zeta_{j,p}^{\alpha-1})}{R^{2\alpha}}\right)$$
$$=\frac{\alpha}{\pi R^{\alpha}}\int_{-\pi/(2\alpha)}^{\pi/(2\alpha)}\log|f_p(Re^{i\theta})|\cos\alpha\theta\,d\theta$$
$$+\frac{\alpha}{2\pi}\int_{\varepsilon}^{R}\left(\frac{1}{t^{2\alpha}}-\frac{1}{R^{2\alpha}}\right)\log|f_p(te^{i\pi/(2\alpha)})f_p(te^{i\pi/(2\alpha)})|\,dt$$
$$+\frac{\alpha}{2\pi}\int_{-\pi/(2\alpha)}^{\pi/(2\alpha)}\Re\left[\log f_p(\varepsilon e^{i\theta})\left(\frac{e^{-i\alpha\theta}}{\varepsilon^{\alpha}}-\frac{\varepsilon^{\alpha}e^{i\alpha\theta}}{R^{2\alpha}}\right)\right]d\theta.$$

The rest of the proof differs from that of Theorem 1 only by routine technical details.

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